定理 8.1(余因子展開)　n 次正方行列 $A = (a_{ij})$ とその余因子について

$$a_{i1}\tilde{a}_{j1} + a_{i2}\tilde{a}_{j2} + \cdots + a_{in}\tilde{a}_{jn} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$a_{1i}\tilde{a}_{1j} + a_{2i}\tilde{a}_{2j} + \cdots + a_{ni}\tilde{a}_{nj} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

が成り立つ．

(証明) 次の二つの場合を示せば十分であろう．

$$a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + \cdots + a_{1n}\tilde{a}_{1n} = \det A$$

$$a_{11}\tilde{a}_{21} + a_{12}\tilde{a}_{22} + \cdots + a_{1n}\tilde{a}_{2n} = 0$$

他の場合については，行と行の入れ替えや転置行列を考えればよい．
まず，$S_n^{(1)} = \{\, \sigma \in S_n \,|\, \sigma(1) = 1 \,\}$ とおくと

$$\tilde{a}_{11} = \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n^{(1)}} \operatorname{sgn}(\sigma) a_{2\sigma(2)} a_{2\sigma(3)} \cdots a_{n\sigma(n)}$$

であることに注意する．これより

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n^{(1)}} \operatorname{sgn}(\sigma) a_{11} a_{2\sigma(2)} \cdots a_{n\sigma(n)} = a_{11}\tilde{a}_{11}$$

がわかり，さらに，$j = 1, 2, \ldots, n$ について，隣り合う列の入れ替えを繰り返し行うことにより，

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1j} & 0 & \cdots & 0 \\ a_{21} & \cdots & a_{2(j-1)} & a_{2j} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

$$= (-1)^{j-1} \begin{vmatrix} a_{1j} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{2j} & a_{21} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{nj} & a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{1j}(-1)^{j-1} \begin{vmatrix} a_{21} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix} = a_{1j}\tilde{a}_{1j}$$

が得られる．

よって，第 1 行に関する線型性により

$$
\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&= \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ a_{21} & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix} \\
&= a_{11}\tilde{a}_{11} + a_{12}\tilde{a}_{12} + \cdots + a_{1n}\tilde{a}_{1n}
\end{aligned}
$$

となることがわかる．$a_{11}\tilde{a}_{21} + a_{12}\tilde{a}_{22} + \cdots + a_{1n}\tilde{a}_{2n}$ については，今の議論を逆に辿ることにより

$$
\begin{aligned}
&a_{11}\tilde{a}_{21} + a_{12}\tilde{a}_{22} + \cdots + a_{1n}\tilde{a}_{2n} \\
&= -a_{11}\begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} + a_{12}\begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots + (-1)^n a_{1n}\begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} \\ a_{31} & \cdots & a_{3(n-1)} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} \end{vmatrix} \\
&= -\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} - \begin{vmatrix} 0 & a_{12} & 0 & \cdots & 0 \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} - \cdots - \begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{31} & \cdots & a_{3(n-1)} & a_{3n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix} \\
&= -\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}
\end{aligned}
$$

となり，第 1 行と第 2 行が一致するので，この行列式の値は 0 である．